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A NON-ANALYTIC PROOF OF THE NEWMAN—ZNÁM RESULT FOR DISJOINT COVERING SYSTEMS

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A direct combinatorial proof is given to a generalization of the fact that the largest modulus N of a disjoint covering system appears at least p times in the system, where p is the smallest prime dividing N. The method is based on geometric properties of lattice parallelotopes.

1. Introduction

A *disjoint covering system* γ is a partition of the integers into residue sets $R_1, \ldots, R_t$, with $t \geqq 2$, of the form $R_i = \{k \in \mathbf{Z} : k \equiv a_i \pmod{N_i}\}$ $(i \in \{1, \ldots, t\})$. The positive integer N_i is the *modulus* of R_i $(1 \leqq i \leqq t)$. Without loss of generality we may assume $N_1 \leqq \ldots \leqq N_t$. The a_i are arbitrary integers. The multiset of moduli is denoted by $\mathcal{N}^{(\gamma)} = \{N_1, \ldots, N_t\}$. Mirsky, D. Newman, Davenport and Rado gave, over 30 years ago, an ingenious proof, using a generating function and roots of unity, that if γ is a disjoint covering system, then $N_{t-1} = N_t$ (see Erdős [2]). Porubský [4, Sect. 2.1] remarked, "No proof of this result is known which does not use complex numbers".

For any integer $k \geqq 2$, denote by $p(k)$ the smallest prime divisor of k: $p(k) = \min\{p \in \mathrm{N} : p \geqq 2,\ p|k,\ p \text{ prime}\}$. M. Newman [3] and Znám [5] generalized the above result by proving $N_{t-p(N_t)+1} = \ldots = N_t$. These proofs also use generating functions and roots of unity.

Our main result is an elementary proof of the Newman—Znám theorem. The proof is based on geometric properties of lattice parallelotopes. This approach can be used to prove other results in the area of covering systems. See Berger, Felzenbaum and Fraenkel [1] for some of these.

A modulus $N \in \mathcal{N}^{(\gamma)}$ is called *division maximal* (in short: *divmax*) if $N_i \in \mathcal{N}^{(\gamma)}$ and $N|N_i$ imply $N_i = N$. In particular, the largest modulus $\mathcal{N}^{(\gamma)}$ is divmax.

We prove,

Theorem. *Let γ be a disjoint covering system, and let N be a divmax element of $\mathcal{N}^{(\gamma)}$. Then N is repeated at least $p(N)$ times in $\mathcal{N}^{(\gamma)}$.*

In Section 2 we develop the necessary geometric tools, and in Section 3 we apply them to the case of a coset partition of a cyclic group. Viewing a disjoint covering system as a coset partition of a cyclic group, this application produces a proof of the theorem.

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2. Lattice parallelotopes

For $\mathbf{b}=(b_1, \ldots, b_n)\in\mathbf{Z}^n$ with $b_i\geqq 2$ $(1\leqq i\leqq n)$, define the *lattice parallelotope* or simply *parallelotope*

$$P = P(n; \mathbf{b}) = \{\mathbf{c} = (c_1, \ldots, c_n)\in\mathbf{Z}^n : 0 \leqq c_i < b_i \quad (1 \leqq i \leqq n)\}$$
$$= B_1\times B_2\times\ldots\times B_n,$$

where $B_i=\{0, \ldots, b_i-1\}$ $(1\leqq i\leqq n)$. If $b_1=\ldots=b_n=b$, then $P(n; \mathbf{b})$ is also called the *cube* $U(n; b)$.

Definition 1. Given a parallelotope $P=P(n; \mathbf{b})$, let $I\subseteqq\{1, \ldots, n\}$. An *$I(K)$-cell*, *$I$-cell* or simply *cell* K of P is a set of the form

$$K = \{\mathbf{s} = (s_1 \ldots, s_n)\in\mathbf{Z}^n : 0 \leqq s_i < b_i \quad \text{for} \quad i\in I, \quad s_i = u_i \quad \text{for} \quad i\notin I\}$$
$$= D_1\times D_2\times\ldots\times D_n,$$

where $D_i=\{0, \ldots, b_i-1\}$ for $i\in I$, $D_i=\{u_i\}$ for $i\notin I$. Here $\mathbf{u}=(u_1, \ldots, u_n)$ is an arbitrary point in P. The set $I=I(K)$ is called the *index* of K.

Let S be any set. As usual we denote by $|S|$ the cardinality of S. If S is a set of lattice points, then $|S|$, the number of lattice points in S, is also called the *volume* of S.

Remarks. 1. The index of a cell K is the set of axes on which the projection of K has the same full length as P. On any axis j outside $I(K)$, the projection of K contains a single point, $u_j\in[0, b_j-1]$.

2. Any cell K determines its index uniquely. This follows from the preceding remark and the fact that the projection of P on any axis $i\in I(K)$ contains at least two points, since $b_i\geqq 2$.

Example 1. Let $P=P(4; 2, 3, 4, 5)$. Here and in some of the following examples, we display in the plane higher-dimensional parallelotopes as a cartesian product of pairs of axes (and one single axis if the dimension is odd). The numbering of axes is indicated in Figure 1 and in the sequel by numbers in parentheses.

Let $\mathbf{u}=(1, 2, 0, 3)\in P$ and $I_1=\{1, 4\}$. Then the encircled points in Figure 1 constitute an I_1-cell K_1. Letting $I_2=\{3\}$, the points encased in squares constitute an I_2-cell.

Definition 2. A partition τ of a parallelotope P into cells is called a *cell partition* of P. A cell $K\in\tau$ is said to be *subset minimal* (in short: *submin*) if $K_i\in\tau$, $I(K_i)\subseteqq I(K)\Rightarrow I(K_i)=I(K)$.

Example 2. Given the parallelotope $P(4; 2, 2, 2, 3)$, a partition τ of P into cells $K_i (1\leqq i\leqq 8)$ is indicated in Figure 2, where the cells K_i are identified by means of their subscripts i in the center of the circles. Incidentally, by Remark 2, we can write down the indices of K_i: $I(K_i)=\{1, 2\}$ for $i=1, 2$; $I(K_i)=\{4\}$ for $i=3, 4, 5, 6$; $I(K_i)=\{1\}$ for $i=7, 8$. The cells K_1 and K_2 are not submin, but K_7 and K_8 are. Also K_i is submin for $3\leqq i\leqq 6$.

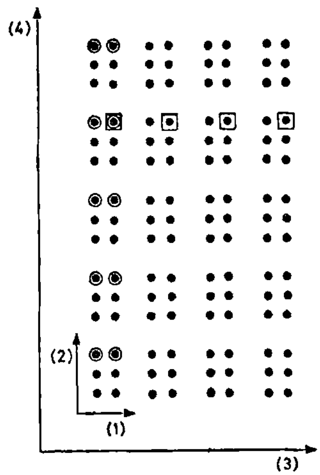

Fig. 1. The parallelotope $P(4; 2, 3, 4, 5)$ with two cells

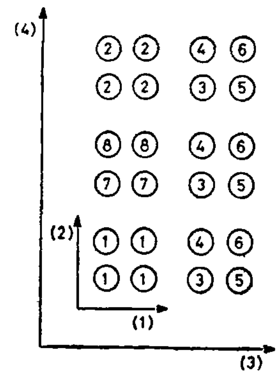

Fig. 2. A cell-partition τ of $P(4; 2, 2, 2, 3)$

Some basic properties of cells will be established now.

Lemma 1. *Given a parallelotope* $P(n; \mathbf{b})$ *and two cells* K_1 *and* K_2 *of* P.
(i) *If* $K_1 \cap K_2 \neq \emptyset$ *then* $L = K_1 \cap K_2$ *is a cell of* P *with index* $I(L) = I(K_1) \cap I(K_2)$.
(ii) *If* $I(K_1) \cup I(K_2) = \{1, \dots, n\}$ *and* $I(K_1) \cap I(K_2) = \emptyset$, *then* $K_1 \cap K_2$ *is a singleton.*

Example 3. Consider the cube $U(3; 2)$ and cells K_1, K_2, K_3 (Figure 3) with $I(K_1) = \{1, 3\}$, $I(K_2) = \{2, 3\}$, $I(K_3) = \{2\}$. We note that $K_1 \cap K_2$ is the cell $L = (1) \times (1) \times P(1; 2)$ with $I(L) = I(K_1) \cap I(K_2) = \{3\}$. Also, $I(K_1) \cup I(K_3) = \{1, 2, 3\}$, $I(K_1) \cap I(K_3) = \emptyset$, and $K_1 \cap K_3$ is the singleton $(0, 1, 0)$.

Proof. Write $K_1 = A_1 \times \dots \times A_n$, $K_2 = B_1 \times \dots \times B_n$, where

$$A_i = \begin{cases} \{0, \dots, b_i - 1\} & \text{if } i \in I(K_1) \\ \text{singleton} & \text{if } i \notin I(K_1), \end{cases}$$

$$B_i = \begin{cases} \{0, \dots, b_i - 1\} & \text{if } i \in I(K_2) \\ \text{singleton} & \text{if } i \notin I(K_2) \quad (1 \leq i \leq n). \end{cases}$$

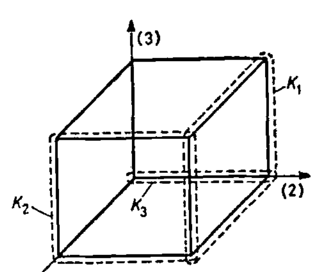

Fig. 3. The cube $U(3; 2)$ with cells K_1, K_2, K_3, illustrating Lemma 1

Now $L=K_1\cap K_2=(A_1\cap B_1)\times\ldots\times(A_n\cap B_n)$. Then $A_i\cap B_i=\{0,\ldots,b_i-1\}$ if $i\in\in I(K_1)\cap I(K_2)$. Assume $K_1\cap K_2\neq\emptyset$. Every point $p\in K_1\cap K_2$ projects onto a point on every axis. Hence $A_i\cap B_i$ is a singleton if $i\notin I(K_1)$ or $i\notin I(K_2)$. Thus L is a cell with $I(L)=I(K_1)\cap I(K_2)$.

For the second part, note that $A_i\cap B_i$ is a singleton for every $i\in\{1,\ldots,n\}$. Hence $K_1\cap K_2$ is a singleton. ∎

Our main result in this section is the following

Lemma 2. *Let τ be a cell-partition of $P(n;\mathbf{b})$ into at least two cells, and let $E\in\tau$ be subset minimal. Put $b=\min\{b_i: i\notin I(E)\}$. Then τ contains at least b $I(E)$-cells.*

Example 4. Letting $E=K_7$ in Example 2, we have $I(K_7)=\{1\}$, $b=2$. In fact, τ contains precisely two I-cells, K_7 and K_8. If we let $E=K_3$, then $I(K_3)=\{4\}$, $b=2$, and τ contains four I-cells, K_3, K_4, K_5, K_6.

Proof. *Case I.* We first consider the special case $I(E)=\emptyset$, so $E=(u_1,\ldots,u_n)$ is a singleton. By taking an interval $L_i\subseteq[0,b_i-1]$ of length $|L_i|=b$ with $u_i\in L_i$ for every $i\in\{1,\ldots,n\}$, we see that there exists a translation $L=L_1\times\ldots\times L_n$ of the cube $U(n;b)$ containing E. Observe that Lemma 1 (i) implies that $\tau_1=\{K\cap L: K\in\tau,\ K\cap L\neq\emptyset\}$ is a translated cell-partiton of L, and that each singleton in τ_1 is also contained in τ.

The volume of any cell in τ_1 is a nonnegative power of b, and the volume of L is b^n. Since τ_1 is a cell-partition of L, the number of singletons in τ_1 must be a multiple of b. Since τ_1 contains a singleton E, τ_1 must in fact contain a positive multiple of b singletons.

Case II. Now $I(E)\neq\emptyset$. By possibly renaming axes, we may assume that $I(E)=\{k+1,\ldots,n\}$ for some $k\in\{1,\ldots,n-1\}$. Consider the cell $D=P_1(k;b_1,\ldots,b_k)\times(0,\ldots,0)$ ($n-k$ zeros) of P, which is an isomorphic copy of the k-dimensional parallelotope $P_1(k;b_1,\ldots,b_k)$. Note that D is orthogonal to E and $I(D)==\{1,\ldots,k\}$.

We claim that on account of the minimality of E, τ induces a cell-partition $\tau_1=\{K\cap D: K\in\tau,\ K\cap D\neq\emptyset\}$ of D, such that a cell K in τ induces a singleton in τ_1 if and only if K is an $I(E)$-cell. Since P_1 is a parallelotope and an isomorphic copy of D, Case I above implies that the number of singletons in τ_1 is at least $b==\min\{b_i: 1\leq i\leq k\}$, that is, τ contains at least $b=\min\{b_i: i\notin I(E)\}I(E)$-cells.

Example 5. Consider $P(3;2,3,2)$ and a cell-partition τ into five cells as depicted in Figure 4a. The cells E_1 and E_2 are submin (with $I(E_1)=I(E_2)=\{3\}$) and orthogonal to $D=P_1(2;2,3)\times(0)$. In Figure 4b we see the induced cell-partition τ_1 of τ, in which precisely the cells E_1 and E_2 correspond to singletons.

To prove the claim, let $K\in\tau$. Assume that $K\cap D$ is a singleton. By Lemma 1 (i), $I(K\cap D)=\emptyset=I(K)\cap I(D)$. Hence $I(K)\subseteq I(E)$. Since E is submin, $I(K)=I(E)$. Conversely, suppose $I(K)=I(E)$. Then $I(K)\cap I(D)=\emptyset$, so $K\cap D$ is a singleton by Lemma 1 (ii), proving the claim.

Thus τ_1, which is a cell-partition of D by Lemma 1 (i), has precisely as many singletons as there are $I(E)$-cells in τ. Since τ_1 can also be considered a cell-partition of P_1, the result follows. ∎

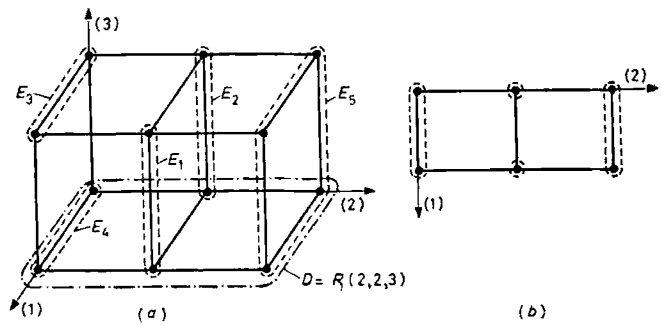

Fig. 4. (*a*) A cell-partition τ of $P(3; 2, 3, 2)$ into 5 cells;
(*b*) The induced cell-partition τ_1 of $P_1(2; 2, 3)$

3. A group to parallelotope mapping

In this Section we establish a connection between cyclic groups and lattice parallelotopes. Using this and an obvious connection between cyclic groups and disjoint covering systems, we will be able to use Lemma 2 to prove the theorem.

Let $\sigma=\sigma_M=\{0, \dots, M-1\}$ be the cyclic group of order M under addition mod M, where $M=p_1^{\alpha_1}\dots p_m^{\alpha_m}$ is the standard form of M (that is, $p_1<\dots<p_m$ primes, $\alpha_1>0, \dots, \alpha_m>0$). Let $k\in\sigma$ and $j\in\{1, \dots, m\}$. Represent the least nonnegative residue k_j of k mod $p_j^{\alpha_j}$ in the p_j-ary numeration system, that is,

$$k_j = \sum_{i=1}^{\alpha_j} a_i^{(j)} p_j^{\alpha_j-i} \equiv k (\operatorname{mod} p_j^{\alpha_j}) \quad (0 \leqq k_j < p_j^{\alpha_j}),$$

where $0\leqq a_i^{(j)}<p_j$ $(1\leqq i\leqq\alpha_j)$.

A *parallelotope function*

$$\Phi\colon \sigma \to P\Big(\sum_{j=1}^{m}\alpha_j;\ \overbrace{p_1, \dots, p_1}^{\alpha_1}, \dots, \overbrace{p_m, \dots, p_m}^{\alpha_m}\Big)$$

is defined as follows. Let $\Phi^{(j)}(k)=(a_1^{(j)}, \dots, a_{\alpha_j}^{(j)})$. Then $\Phi(k)=\big(\Phi^{(1)}(k), \dots, \Phi^{(m)}(k)\big)$.

Example 6. Let $M=24=2^3\times 3$. The p_j-ary representation of the terms in σ_{24} is shown in Table 1 $\big(j\in\{1, 2\}\big)$.

The mapping $\Phi\colon \sigma_{24}\to(4; 2, 2, 2, 3)$ is depicted in Figure 5. We have already met this parallelotope in Figure 2. Note that in the latter, "monochromatic cells", that is, collections of circles labeled with the same number, correspond to cosets of subgroups, as can be seen from Figure 5. This is not an accident, as will be shown in Lemma 3 below.

Define an addition operation on parallelotope lattice points as usual vector addition, where

$$(a_1^{(j)}, \dots, a_{\alpha_j}^{(j)})+(b_1^{(j)}, \dots, b_{\alpha_j}^{(j)}) = (d_1^{(j)}, \dots, d_{\alpha_j}^{(j)}) \quad (1 \leqq j \leqq m)$$

Table 1

The p_j-ary representation for terms in σ_{24}

	$\Phi^{(1)}(k)$			$\Phi^{(2)}(k)$
k	$a_1^{(1)}$	$a_2^{(1)}$	$a_3^{(1)}$	$a_1^{(2)}$
0	0	0	0	0
1	0	0	1	1
2	0	1	0	2
3	0	1	1	0
4	1	0	0	1
5	1	0	1	2
6	1	1	0	0
7	1	1	1	1
8	0	0	0	2
9	0	0	1	0
10	0	1	0	1
11	0	1	1	2
12	1	0	0	0
13	1	0	1	1
14	1	1	0	2
15	1	1	1	0
16	0	0	0	1
17	0	0	1	2
18	0	1	0	0
19	0	1	1	1
20	1	0	0	2
21	1	0	1	0
22	1	1	0	1
23	1	1	1	2

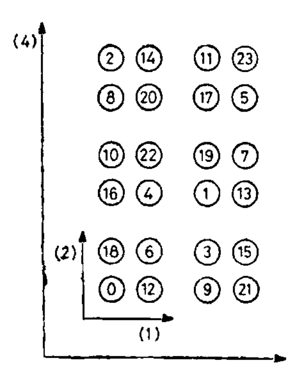

Fig. 5. The mapping
$\Phi: \sigma_{24} \to P(4; 2, 2, 2, 3)$

is the normal p_j- ary addition, with carries propagating from right to left, and any "overflow" digit lost for each $j\in\{1, \ldots, m\}$. Thus

$$0 \leqq d_i < p_j (1 \leqq i \leqq \alpha_j), \quad \sum_{i=1}^{\alpha_j} d_i^{(j)} p_j^{\alpha_j - i} < p_j^{\alpha_j}.$$

Example 7. For σ_{24} and using Table 1, $\Phi(11)+\Phi(23)=(0, 1, 1; 2)+(1, 1, 1; 2)=$ $=(0, 1, 0; 1)$, which, incidentally, is $\Phi(11+23)=\Phi(10)$.

For any sets S, T, function f, real number r and integer d, we use the notation $S+r=r+S$ for the set $\{r+s: s\in S\}$; the notation $f(S)$ for $\{f(s): s\in S\}$; we write $S\equiv T \pmod d$ if $|S|=|T|$, for every $s\in S$ there is $t\in T$ such that $s\equiv t \pmod d$, and for every $t\in T$ there is $s\in S$ such that $t\equiv s \pmod d$.

Lemma 3. (i) *The mapping Φ is bijective.*

(ii) *The mapping Φ is additive: For $k, l\in\sigma$, $\Phi(k+l)=\Phi(k)+\Phi(l)$.*

(iii) *If C is any coset of any subgroup H of σ with $|H|=p_1^{\beta_1}\ldots p_m^{\beta_m}$, $0\leqq\beta_i\leqq\alpha_i$ $(1\leqq i\leqq m)$, then $\Phi(C)$ is a cell of P with index*

$$I(C) = \bigcup_{j=1}^{m} \left(\{1, \ldots, \beta_j\} + \sum_{i=1}^{j-1} \alpha_i\right)$$

(which depends only on $|C|$, not on C), and volume $|\Phi(C)|=|C|=|H|$.

We note in passing that if β_j of the α_j axes in the interval $\left[\sum_{i=1}^{j-1}\alpha_i, \sum_{i=1}^{j}\alpha_i-1\right]$ lie in the index, then it is the *first* β_j in the interval which are in the index.

Proof. (i) Suppose $\Phi(k)=\Phi(h)$ for some $k, h\in\sigma$, that is, $\Phi^{(j)}(k)=\Phi^{(j)}(h)$ for $1\leqq j\leqq m$. By the uniqueness of p-ary representations, this implies $k_j=h_j$ $(1\leqq j\leqq m)$. The Chinese Remainder Theorem then implies $k=h$.

(ii) Note that lattice point addition transforms the set

$$\{(\Phi^{(1)}(k), \ldots, \Phi^{(m)}(k)): k\in\sigma\}$$

into a cyclic group of order M, which is thus isomorphic to σ. Hence $\Phi(k+l)=$ $=\Phi(k)+\Phi(l)$.

(iii) We have $C=\{kM/|C|: 0\leqq k<|C|\}+d$, for some $d\in[0, (M/|C|)-1]$. Let $j\in\{1, \ldots, m\}$. For simplicity we often write p, α, β for p_j, α_j, β_j below. Now

$$\frac{M}{|C|} = p_1^{\alpha_1-\beta_1} \ldots p_m^{\alpha_m-\beta_m} \equiv cp^{\alpha-\beta} \pmod{p^\alpha}, \quad (c, p) = 1.$$

Since clearly $kcp^{\alpha-\beta}\equiv(k+p^\beta)cp^{\alpha-\beta} \pmod{p^\alpha}$, we have $C\equiv\{kcp^{\alpha-\beta}: 0\leqq k<p^\beta\}+d_j$ $\pmod{p^\alpha}$, where d_j is the least nonnegative residue of d mod p^α. Since $(c, p)=1$, we clearly have $\{kc: 0\leqq k<p^\beta\}\equiv\{l: 0\leqq l<p^\beta\} \pmod{p^\beta}$. Multiplying by $p^{\alpha-\beta}$ we get finally, $C\equiv\{lp^{\alpha-\beta}: 0\leqq l<p^\beta\}+d_j \pmod{p^\alpha}$, where $lp^{\alpha-\beta}<p^\alpha$ for every $l\in[0, p^\beta-1]$.

If $l=\sum_{i=1}^{\beta} a_i p^{\beta-i}$ is the p-ary representation of l, then $lp^{\alpha-\beta}=\sum_{i=1}^{\beta} a_i p^{\alpha-i}$, so

$$\Phi^{(j)}(H) = \{(a_1, \ldots, a_\beta, \overbrace{0, \ldots, 0}^{\alpha-\beta}): 0 \leqq a_i < p (1 \leqq i \leqq \beta)\}.$$

By part (ii),

$$\Phi^{(j)}(C) = \Phi^{(j)}(H)+\Phi^{(j)}(d_j) =$$
$$= \{(a_1, \dots, a_\beta, 0, \dots, 0)\colon 0 \leqq a_i < p \quad (1 \leqq i \leqq \beta)\}+\Phi^{(j)}(d_j).$$

This shows that in the interval $[\sum_{i=1}^{j-1} \alpha_i, \sum_{i=1}^{j} \alpha_i - 1]$ precisely the axes $\{1, \dots, \beta_j\}+ + \sum_{i=1}^{j-1} n_i$ belong to the index. Hence C has index

$$\bigcup_{j=1}^{m} (\{1, \dots, \beta_j\}+ \sum_{i=1}^{j-1} \alpha_i)$$

as claimed. Finally, $|\Phi(C)|=|C|$ since Φ is a bijection. ∎

Proof of Theorem. Let $M=\text{l.c.m.}(N_1, \dots, N_t)=p_1^{\alpha_1}\dots p_m^{\alpha_m}$ be the standard form of M. With the set $R_i=\{k\in\mathbf{Z}\colon k\equiv a_i \,(\text{mod}\, N_t)\}$ of γ, associate the coset $C_i= =\{nN_i+a_i\colon nN_i+a_i\in\sigma_M\}$ of the subgroup $H_i=\{nN_i\colon 0\leqq n<M/N_i\}$ of $\sigma=\sigma_M$. Then $|C_i|=|H_i|=M/N_i$, and $\{C_1, \dots, C_t\}$ is a coset partition γ' of σ, that is, a partition of σ into cosets of subgroups of σ. Let $C=\{nN+a\colon nN+a\in\sigma\}$, where N is a divmax element of $\mathcal{N}^{(\gamma)}$. Since $N|C|=N_i|C_i|=M$ $(1\leqq i\leqq t)$, we have that $|C_i|\,\big|\,|C|$ implies $|C_i|=|C|$.

Consider the parallelotope function

$$\Phi\colon \sigma \to P\Big(\sum_{i=1}^{m} \alpha_j, \overbrace{p_1, \dots, p_1}^{\alpha_1}, \dots, \overbrace{p_m, \dots, p_m}^{\alpha_m}\Big).$$

By Lemma 3, $\Phi(\gamma')$ is a cell-partition τ of P. Let $\Phi(C)=K$, $\Phi(C_i)=K_i$ $(1\leqq i\leqq m)$.

Suppose $I(K_i)\subseteqq I(K)$ for some i. Since the volume of a cell is obviously the product of the lengths of its projections on all the axes of the index, we have $|K_i|\,\big|\,|K|$. By the last part of Lemma 3, this is equivalent to $|C_i|\,\big|\,|C|$. Thus

$$|K_i|\,\big|\,|K| \Rightarrow |C_i|\,\big|\,|C| \Rightarrow |C_i| = |C| \Rightarrow |K_i| = |K| \Rightarrow I(K_i) = I(K),$$

so K is submin.

By Lemma 2, $\Phi(\gamma')$ contains at least b submin cells with the same index as K, where

$$b = \min\Big\{p_j\colon [\sum_{i=1}^{j-1} \alpha_i, \sum_{i=1}^{j} \alpha_i - 1] \subseteqq\hskip-0.9em/\ I(K)\Big\} \quad \text{(by Lemma 3 (iii))}$$
$$= \min\{p_j\colon p_j^{\alpha_j} \nmid |K|\}$$
$$= \min\Big\{p_j\colon p_j \Big| \frac{M}{|K|}\Big\} \qquad \Big(p_j^{\alpha_j} \nmid |K| \;\text{ iff }\; p_j \Big| \frac{M}{|K|}\Big)$$
$$= \min\{p_j\colon p_j|N\} \qquad (M/|K| = N)$$
$$= p(N). \quad ∎$$

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